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# Random Walks in Space Time Mixing Environments

**Jean Bricmont · Antti Kupiainen**

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**Abstract** We prove that random walks in random environments, that are exponentially mixing in space and time, are almost surely diffusive, in the sense that their scaling limit is given by the Wiener measure.

**Keywords** Renormalization group · Diffusion · Scaling limit

## 1 The Results

Random walks in random environments are walks where the transition probabilities are themselves random variables (see [22, 24] for recent reviews of the literature). The environments can be divided into two main classes: static and dynamical ones. In the first case, the transition probabilities are given once and for all, and the walk can be "trapped" for a long time in some regions because the transition probabilities happen to favour motion towards that region. This may lead to anomalously slow diffusion in one dimension, as was shown by Sinai [21]. In [2, 23], it is shown that, in three or more dimensions and for weak disorder (almost deterministic walks), ordinary diffusion takes place.

In dynamical environments, the random transition probabilities change with time and trapping does not occur, so that one expects ordinary diffusion to hold in all dimensions. Although simpler than the static environments, the dynamical ones are not trivial to analyze; see [12, 13] for recent and general results and for references to earlier ones.

We consider in this paper a rather general class of space-time mixing environments. This means that the transition probabilities at different times and spatial points are weakly correlated and moreover the randomness is weak. For such environments we prove that the

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J. Bricmont
FYMA, UCL, B chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium

A. Kupiainen (✉)
Department of Mathematics, Helsinki University, P.O. Box 4, 00014 Helsinki, Finland
e-mail: ajkupiai@cc.helsinki.fi

walks are diffusive, almost surely in the environment measure. In particular we do not assume a Markovian structure of the environment. We only assume that certain cumulants (or connected correlation functions) decay in a way that is typical of what happens in high temperature or weakly coupled Gibbs states.

Our motivation to study this class of models comes from the consideration of random walks in a deterministic, but "chaotic" environment [11]. As shown first by Bunimovich and Sinai, the invariant measures of suitably coupled hyperbolic dynamical systems correspond, via an extension of the SRB formalism, to certain weakly coupled Gibbs states for a spin system on a space-time lattice [3–5, 8, 15–17]. A walk whose transition probabilities are local functions of such hyperbolic systems can be analyzed by the methods developed here.

Random walks in such deterministic environments emerge when considering deterministic dynamics of a coupled map lattice with a global conserved quantity ("energy"). The latter in turn can be viewed as a model of coupled Hamiltonian systems where one would like to prove diffusion and Fourier's law for heat transport. In such models the environments will have more general correlations than the Markovian ones and we expect to use the method developed in this paper. This is discussed further at the end of this section and in [6].

The method used in the proof consists in applying a Renormalization group scheme to iterate bounds, both on the size of the coupling between the transition probabilities, and on the size of their "disorder", i.e. of their deviation from a deterministic walk. In the long time limit, the disorder tends to zero and the resulting deterministic walk behaves diffusively.

Turning to the precise models considered here, let $\Omega^{\mathbf{T}}$ be the space of walks $\omega = (\omega_0, \dots, \omega_{\mathbf{T}})$, $\omega_t \in \mathbb{Z}^d$, in time $\mathbf{T}$ and starting at $\omega_0 = 0$ and let the probability of a walk be defined as

$$P^{\mathbf{T}}(\omega) = \prod_{t=0}^{\mathbf{T}-1} p(t, \omega_t, \omega_{t+1}). \tag{1.1}$$

The transition probabilities $p(t, u, v)$ of the walk are taken to be random variables defined on some probability space $\Xi$, with distribution $\mathcal{P}$, satisfying the following assumptions:

A.1. *Probability* $p(t, u, v) \ge 0$ and $\sum_v p(t, u, v) = 1$.

A.2. *Homogeneity and isotropy*. Let $\tau_s$, $\tau_w$ denote translations in time and space. We assume that $\tau_s \tau_w p$ has the same law as $p$. For $R$ a rotation around the origin fixing the lattice $\mathbb{Z}^d$ we assume that $p(t, u, v)$ and $p(t, u, u + R(v - u))$ are identically distributed for all $t, u, v$.

A.3. *Weak randomness*. Let $\langle - \rangle$ denote the expectation with respect to $\mathcal{P}$ and define

$$T(u - v) := \langle p(t, u, v) \rangle \tag{1.2}$$

$$b(t, u, v) := p(t, u, v) - T(u - v). \tag{1.3}$$

(where translation invariance was used). Let, for $k \in \mathbb{T}^d$,

$$\hat{T}(k) = \sum_{u \in \mathbb{Z}^d} \exp(-iku) T(u), \tag{1.4}$$

be the Fourier transform of $T$. We assume that $\hat{T}$ is analytic in a complex neighborhood of $\mathbb{T}^d$ with

$$\hat{T}(k) = 1 - ck^2 + \mathcal{O}(|k|^4) \tag{1.5}$$

in a neighborhood of origin where $c>0$ and

$$|\hat{T}(k)|<1 \tag{1.6}$$

for $k\in\mathbb{T}^d\setminus 0$.

About the "random" part $b$, we will assume that it has small correlation functions decaying exponentially in space and time as specified in (1.11) below.

*Remark 1* Analyticity implies that $T(u)$ is exponentially decaying. Note that for the transition matrix of nearest neighbour random walks, $\hat{T}(k)=\frac{1}{d}\sum_{j=1}^d \cos k_j$, which does not satisfy (1.6) at $k_j=\pi$, $\forall j$. However, if we take for $T$ the previous transition matrix composed with itself (i.e. nearest neighbour random walks after two steps), we get $\hat{T}(k)=(\frac{1}{d}\sum_{j=1}^d \cos\frac{k_j}{2})^2$, and (1.6) holds (see [2], Sect. 5, for a discussion of this point).

We now explain the assumptions made on the random matrices $b$. We denote the pair $u,v$ by $z$ and $b(t,u,v)$ by $b(t,z)$. Given $A\subset\mathbb{Z}$, introduce variables $z_t$ for $t\in A$ and define

$$b_A(z):=\prod_{t\in A} b(t,z_t). \tag{1.7}$$

Since we need to deal with expectations of (1.7) with possibly several copies of the same $b(t,z)$ we extend the definition (1.7) to the disjoint union

$$A=\coprod_{i=1}^m A_i \tag{1.8}$$

of $A_i\subset\mathbb{N}$, $i=1,\dots,m$.

Recall the definition of the connected correlation functions (or *cumulants*)

$$\langle b_A\rangle^c=\sum_{\Pi\in\mathcal{P}(A)}(-1)^{|\Pi|+1}\prod_{B\in\Pi}\langle b_B\rangle, \tag{1.9}$$

where $\mathcal{P}(A)$ is the set of partitions of $A$.

We assume that these cumulants decay exponentially in the temporal and spatial separations in (1.9). To spell this out let, for $B\subset\mathbb{N}$, $d(B)$ be the diameter of $B$, and for $A$ as in (1.8) $d(A)=d(\cup A_i)$.

For the spatial dependence, let, for a finite set $S\subset\mathbb{R}^d$, $\tau(S)$ be the length of the shortest connected graph whose vertices are a subset of $\mathbb{R}^d$ containing $S$. For $A$ and $z$ as above, define

$$\tau_A(z):=\sum_{t\in A}|u_t-v_t|+\tau(S(z)), \tag{1.10}$$

where $S(z)$ is the set of $u_t$ and $v_t$ in $z$ (the reasons why we need this definition for $S\subset\mathbb{R}^d$ instead of simply $S\subset\mathbb{Z}^d$ will be clear in the next section).

We assume that:

$$\|\langle b_A\rangle^c\|:=\sup_z e^{\lambda\tau_A(z)}|\langle b_A(z)\rangle^c|\le\epsilon^{|A|}e^{-\lambda d(A)}, \tag{1.11}$$

for all $A$ of the form (1.8) with $m\le n_0$, $\epsilon$ small enough and $\lambda$ large enough. Here, $|A|=\sum_{i=1}^m|A_i|$.

We will study in this paper the large $\mathbf{T}$ properties of the probability measure on paths defined by (1.1). It will be convenient to realize them as measures $\nu_{\mathbf{T}}$ on $C([0,1])$, the space of continuous paths $\omega: [0,1] \to \mathbb{R}^d$, by rescaling the time in a standard way. Thus, given an $\omega \in \Omega$, we obtain a piecewise linear path

$$\omega(t) = \mathbf{T}^{-\frac{1}{2}}(\omega_{i-1} + (\mathbf{T}t - i + 1)(\omega_i - \omega_{i-1})), \tag{1.12}$$

where $i - 1 = [\mathbf{T}t]$ and [ ] denotes the integral part. $\nu_{\mathbf{T}}$ is the measure (1.1), transposed by (1.12), on $C([0,1])$, and we will study the limit $\lim_{\mathbf{T}\to\infty} \nu_{\mathbf{T}}$, also called the *scaling limit*, and its properties. For reasons of convenience that will be explained in the next section, we will consider below times of the form $\mathbf{T} = L^{2n}$ for $n \in \mathbb{N}$ and $L$ a fixed integer chosen later. We will denote $\nu_{L^{2n}}$ by $\nu_n$ for short and expectations in $\nu_n$ by $\mathcal{E}_n$. We let similarly $E_{\mathbf{T}}$ (or $E_n$) refer to expectation in $P^{\mathbf{T}}$. They are related simply by

$$\mathcal{E}_n F(\omega(\cdot)) = E_n F(L^{-n}\omega_{L^{2n}\cdot}), \tag{1.13}$$

for functions $F$ depending on $\omega$ restricted to $L^{-2n}\mathbb{Z}$.

We now state the main result concerning the scaling limit. Let $\nu^D$ be the Wiener measure with diffusion constant $D$ on paths $\omega \in C([0,1])$ with $\omega(0) = 0$ and $\mathcal{E}^D$ be the corresponding expectation. The scaling limit of our walk is given by $\nu^D$ for almost all environments. We prove that suitable correlation functions converge, and this implies convergence of the diffusion constant and of the finite dimensional distributions (take $f(x) = e^{ikx}$ below, and use Theorem 7.6 in [1]).

**Theorem 1** *Let $\mathcal{P}$ satisfy* A.1–A.3. *Then there is an $\epsilon_0 > 0$ and $\lambda_0$ such that, for $\epsilon < \epsilon_0$, $\lambda > \lambda_0$ in* (1.11), *there exists a $D > 0$ such that, for any family $f_1 \dots f_\kappa$, of polynomially bounded continuous functions, and $t_1 \dots t_\kappa \in [0,1]$,*

$$\lim_{n\to\infty} \mathcal{E}_n \prod_i f_i(\omega(t_i)) = \mathcal{E}^D \prod_i f_i(\omega(t_i))$$

*$\mathcal{P}$-almost surely.*

*Remark 2* The diffusion constant $D$ *satisfies* (see (2.21))

$$|D - D_0| \le C\epsilon^2, \tag{1.14}$$

where

$$D_0 = \sum_{u\in\mathbb{Z}^d} T(u)u^2. \tag{1.15}$$

*Remark 3* In Theorem 1, we prove convergence of finite dimensional distributions, and only along a subsequence of times. However, with extra work $\mathcal{P}$-almost sure weak convergence (for the full sequence of times) also follows. We sketch how to do that after the proof of Theorem 1 in the last section.

*Remark 4* Also with some more work, one should still be able to obtain Theorem 1 while replacing $\tau(S(z))$ in the definition (1.10) of $\tau_A(z)$ by $\mathrm{diam}(S(z))$. Indeed, the main point where the decay in $\tau_A(z)$ (see (1.11)) is used, is to control the integral (3.36) below. This should then allow an extension of the result of Example 2 below to the coupled map lattices considered in [4], with smooth maps instead of analytical ones.

Let us finally give examples satisfying our assumptions.

*Example 1* Let $\mu$ be the Gibbs measure for a high temperature Ising model on the space time lattice $\mathbb{Z}^{d+1}$ and let $s(t,x)$ be the spins. Let $p(s,x)$ be functions of $x \in \mathbb{Z}^d$ and of the spins $s(t,y)$ for $t, y$ close to 0; let the distribution induced by $\mu$ of $p(\cdot, x)$ be invariant under lattice rotations. Take

$$p(t,u,v) = p(\tau_t \tau_u s, v-u)$$

where $\tau_t$ and $\tau_u$ are translations in time and space. Then $p$ satisfies our assumptions. For a cluster expansion approach to estimates like (1.11), see e.g. [7, 18, 20].

This example generalizes to $p$'s that are local and rotationally invariant functions of the variables distributed by completely analytic Gibbs states (see [9, 10, 14, 19] for definitions and examples of the latter).

*Example 2* As an application of this extension to completely analytic Gibbs states, one may consider, as in [11], a *deterministic environment* generated by a chaotic dynamics. Let $\theta \in \mathcal{M} = \mathbb{T}^{\mathbb{Z}^d}$ and let $f : \mathcal{M} \to \mathcal{M}$ be a *coupled analytic map*, as studied in [3]. Let $\theta(t) = f^t(\theta)$, and

$$p(t,u,v) = p(\tau_u \theta(t), v-u),$$

where $p$ is *local* i.e. depends on $\theta(t,x)$ exponentially weakly in $|x|$. If $p$ is also analytic in $\theta$ and if $\theta$ is distributed by the product of Lebesgue measures on $\mathbb{T}^{\mathbb{Z}^d}$, then one can show, using the cluster expansion in [3], that the assumption (1.11) holds. This example will be discussed further in [6].

## 2 The Renormalization Group

The Renormalization group will allow us to replace the analysis of long time properties of the walk by the study of a map, the Renormalization group map, relating transition probability densities on successive scales.

It will be convenient to extend the transition probabilities $p(t,u,v)$ by constants to unit cubes centered at $u$ and $v$. Then the probability density to go from $u \in \mathbb{R}^d$ to $v \in \mathbb{R}^d$ in the time interval $I = [t,t']$ is given by

$$P_{[t,t']}(u,v,p) = \int d\omega_{t+1} \dots d\omega_{t'-1} \prod_{s=t}^{t'-1} p(s,\omega_s,\omega_{s+1}) \tag{2.1}$$

with $\omega_t = u$, $\omega_{t'} = v$. We stressed in (2.1) the dependence on the random matrix $p$ and below we will use (2.1) also for $p$'s that are not constant on unit cubes.

Let now $l \in \mathbb{N}$ and define a scaled transition probability density

$$R_l p(t,u,v) = l^d P_{[l^2 t, l^2(t+1)]}(lu, lv, p). \tag{2.2}$$

Then, if $l^2$ divides $t, t'$, by a simple change of variables,

$$P_{[t,t']}(u,v,p) = l^{-d} P_{[t/l^2, t'/l^2]}(l^{-1}u, l^{-1}v, R_l p). \tag{2.3}$$

$R_l p$ are the *renormalized transition probability densities* at scale $l$. Note that they are constant on $l^{-1}$ cubes centered at $(l^{-1}\mathbb{Z})^d$. They are functions of $p$ and hence random matrices with a law inherited from $p$. As $l \to \infty$ $R_l p$ controls the long time behavior of the walk. For example, the diffusion constant becomes

$$D(l^2)(p) = l^{-2} \int dy P_{[0,l^2]}(0, y, p)[y]_1^2 = \int dy\ R_l p(0, 0, y)[y]_l^2 = D(1)(R_l p), \tag{2.4}$$

where $[y]_l$ takes the value $x$ at the $l^{-1}$ cube centered at $x \in (l^{-1}\mathbb{Z})^d$. Thus *the long time behavior is reduced to a time* 1 *problem for* $R_l p$*, as* $l \to \infty$.

$R_l$ is called the *renormalization group map*. Obviously it is a semigroup, $R_{ll'} = R_l R_{l'}$ and the large $l$ limit is most conveniently studied iteratively. We choose an integer $L > 1$ and let $R := R_L$ and $p_n = R^n p$ i.e. $p_n = R_{L^n} p$.

To make a connection to the scaling limit, let $F$ in (1.13) depend on $\omega$ restricted to $L^{-2\ell}\mathbb{Z}$ and let $n = \ell + m$. Then, we get from (1.13)

$$\mathcal{E}_n F(\omega(\cdot)) = E_n^p F(L^{-n}\omega_{L^{2n}\cdot}),$$

where we denoted the $p$ dependence explicitly, and then, renormalizing by $l = L^m$,

$$\mathcal{E}_n F(\omega(\cdot)) = E_\ell^{p_m} F(L^{-\ell}\omega_{L^{2\ell}\cdot}). \tag{2.5}$$

This relation will be used to prove Theorem 1.

We will study the iteration

$$p_n \to p_{n+1} = R p_n \tag{2.6}$$

where, from (2.1, 2.2), we have

$$Rp(t, u, v) = L^d \int d\omega_{I_t} \prod_{s \in I_t} p(s, \omega_s, \omega_{s+1}) \tag{2.7}$$

with $I_t = [L^2 t, L^2(t+1) - 1]$, $d\omega_{I_t} = d\omega_{L^2 t+1} \dots d\omega_{L^2(t+1)-1}$ and $\omega_{L^2 t} = Lu, \omega_{L^2(t+1)} = Lv$.

The map $R$ obviously preserves the properties A.1 and A.2, i.e., in particular, $\int dv p_n(t, u, v) = 1$. As for A.3, let us divide $p_n$ into a "deterministic" and a "random" part as in (1.2) and (1.3):

$$p_n(t, u, v) = T_n(u - v) + b_n(t, u, v) \tag{2.8}$$

where

$$T_n(u - v) = \langle p_n(t, u, v)\rangle. \tag{2.9}$$

We have $\int dv T_n(v) = 1$ and thus

$$\int dv b_n(t, u, v) = 0 = \langle b_n(t, u, v)\rangle. \tag{2.10}$$

The bulk of this paper consists in showing that $b_n$ tends a.s. to zero as $n \to \infty$, whereas $T_n$ tends to a Gaussian. The latter claim is evident if $b = 0$. Indeed, for a translation invariant $p$, the RG map (2.2) is just a multiple convolution and becomes in terms of $\hat{T}$, the Fourier transform (1.4) of $T$,

$$\hat{T}_{n+1}(k) = \hat{T}_n\left(\frac{k}{L}\right)^{L^2}, \tag{2.11}$$

i.e.

$$\hat{T}_n(k) = \hat{T}\left(\frac{k}{L^n}\right)^{L^{2n}} := \hat{\mathcal{T}}_n(k). \tag{2.12}$$

By the assumption (1.5) and (1.15)

$$\hat{T}(k) = 1 - (2d)^{-1} D_0 k^2 + \mathcal{O}(|k|^4). \tag{2.13}$$

Hence, as $n \to \infty$, uniformly on compacts,

$$\hat{\mathcal{T}}_n(k) \to e^{-\frac{D_0}{2d}k^2} \equiv \hat{T}^*_{D_0}(k) \tag{2.14}$$

where $T^*_D(x)$ is the unit time transition probability density of the Wiener measure:

$$T^*_D(x) = (2\pi D/d)^{-d/2} e^{-\frac{d}{2D}x^2}. \tag{2.15}$$

Of course $b$ is not zero and, at each scale, $b_n$ will modify the diffusion constant. Since $b_n$ goes to zero, we shall obtain a sequence of approximations $D_n$, see (2.4), to the true diffusion constant $D$.

The renormalization will allow us to iterate the following bounds for $b_n$ and $T_n$. Let

$$\delta_n = L^{-n/2} e^{-\lambda}. \tag{2.16}$$

**Proposition 1** *Under the assumptions of Theorem 1, for all $A$ of the form (1.8)*

$$\| \langle b_{nA} \rangle^c \| \leq C \epsilon^{|A|} \delta_n^{d(A)} \tag{2.17}$$

*and moreover, for $d(A) = 0$, we have*

$$\sup_u \int dv e^{\frac{1}{2}\lambda \tau_A(z)} |\langle b_{nA}(z) \rangle^c| \leq C \epsilon^{|A|} \delta_n. \tag{2.18}$$

As for the deterministic part, we have:

**Proposition 2** *For $n \geq 1$, we have*

$$|T_n(x)| \leq C e^{-|x|}, \tag{2.19}$$

*moreover,*

$$|T_n(x) - T^*_D(x)| \leq C \delta_n e^{-|x|}, \tag{2.20}$$

*where*

$$|D - D_0| \leq C \epsilon^2. \tag{2.21}$$

*Remark 5* (On the choice of constants) In the proofs, we use the letters $c$, $c'$ or $C$ to denote numerical constants independent of $L$ (but that may depend on $\lambda$ and $n_0$) and $c(L)$ or $C(L)$ constants that do depend on $L$. Those constants may vary from place to place, even in the same equation. Since $\lambda$ and $n_0$ are fixed (and in fact, as we'll see in the proof of Theorem 1, $n_0$ could be taken equal to 2), we will usually not indicate the dependence of constants on $\lambda$ or $n_0$. We choose $L$ large enough so that we can always use $C \le L$, or $C \le L^\alpha$ for any given $C$ or $\alpha > 0$ entering into our arguments. And we choose $\epsilon$ small enough so that we can use $C(L)\epsilon \le 1$ for any $C(L)$.

## 3 Linearized RG

From (2.7) and (2.8), dropping the index $n$ and denoting $n+1$ by prime, we have the following recursion relation for $b_n$:

$$b'(t', u', v') = L^d \int d\omega_{I_{t'}} \Big[ \prod_{t \in I_{t'}} (T(\omega_t - \omega_{t+1}) + b(t, \omega_t, \omega_{t+1})) - \langle - \rangle \Big]. \tag{3.1}$$

In this section we will show how the bound in Proposition 1 iterates once the nonlinear relation (3.1) is replaced by its linearization:

$$(\mathcal{L}b)(t', u', v') = L^d \sum_n \int du dv T^n(Lu' - u) T^{L^2-n-1}(Lv', v) b(t, u, v) \tag{3.2}$$

(since $\langle b(t,u,v)\rangle = 0$, there is no subtraction as in (3.1)), where $t = L^2 t' + n$ and $T^0(x) = \delta(x)$ (which takes values $L^{nd}$ on the $L^{-n}$ cube centered at 0, on scale $n$, since the transition probabilities are constant on cubes of side $L^{-n}$).

For each $t' \in A'$ pick $t(t') \in I_{t'}$ and define $n(t') \in [0, L^2 - 1]$ by writing $t(t') = L^2 t' + n(t')$. Let $A$ be the collection of $t(t')$ and let $\mathbf{n}$ be the one of $n(t')$. The linearized RG is then given by

$$\langle (\mathcal{L}b)_{A'}(z')\rangle^c := L^{d|A'|} \sum_{\mathbf{n}} \int du dv M_{\mathbf{n}}(u', u) N_{\mathbf{n}}(v', v) \langle b_A(z)\rangle^c \tag{3.3}$$

where $z = (u, v)$, $z' = (u', v')$ and

$$M_{\mathbf{n}}(u', u) = \prod_{t'} T^{n(t')}(Lu'_{t'} - u_t), \qquad N_{\mathbf{n}}(v', v) = \prod_{t'} T^{L^2-n(t')-1}(Lv'_{t'} - v_t) \tag{3.4}$$

where $t = t(t')$ and the product runs over $t' \in A'$.

In this section, we first prove inductively the bound (2.17) for the linearized part of $b'$, i.e.:

$$\|\langle (\mathcal{L}b)_{A'}\rangle^c\| \le C\epsilon^{|A'|}\delta'^{d(A')}. \tag{3.5}$$

We need first to express the exponent $\tau_{A'}(z')$ in terms of $\tau_A(z)$. Let $G_A$ be a connected graph with a set of vertices including $S(z)$ and of length $\tau(S(z))$. Let $E$ be the graph obtained by joining to $G_A$ the lines with end points $Lu'_{t'}$ and $u_t$ and $Lv'_{t'}$ and $v_t$. Then its length is at least as large as $\tau(S(Lz')) = L\tau(S(z'))$. Hence

$$\tau(S(z')) \le L^{-1}(\tau(S(z)) + \sum_{t'}(|Lu'_{t'} - u_t| + |Lv'_{t'} - v_t|)). \tag{3.6}$$

Since also $|u'_{t'} - v'_{t'}| \le L^{-1}(|Lu'_{t'} - u_t| + |Lv'_{t'} - v_t| + |u_t - v_t|)$ we obtain, using (1.10),

$$\tau_{A'}(z') \le L^{-1}\Big(\tau_A(z) + 2\sum_{t'}(|Lu'_{t'} - u_t| + |Lv'_{t'} - v_t|)\Big). \tag{3.7}$$

Equations (3.3) and (3.7) imply

$$\begin{aligned} I' &:= e^{\lambda\tau_{A'}(z')}|\langle(\mathcal{L}b)_{A'}(z')\rangle^c| \\ &\le L^{d|A'|}\sum_{\mathbf{n}}\int dudv\tilde{M}_{\mathbf{n}}(u',u)\tilde{N}_{\mathbf{n}}(v',v)\cdot e^{\lambda\tau_A(z)/L}|\langle b_A(z)\rangle^c| \end{aligned} \tag{3.8}$$

where $\tilde{M}$ and $\tilde{N}$ are like $M$ and $N$ in (3.4) but with $T^n$ replaced by

$$\tilde{T}^n(u) = T^n(u)e^{\frac{2\lambda}{L}|u|}. \tag{3.9}$$

Let first $d(A') > 1$. Then, since $A$ contains one element in each $I_{t'}$, $t' \in A'$,

$$d(A) \ge L^2(d(A') - 1) \ge \frac{1}{2}L^2 d(A'). \tag{3.10}$$

Since $A'$ contains at least $|A'|/n_0$ distinct times we also have $d(A') \ge |A'|/n_0$. Thus we have

$$d(A) \ge cL^2(|A'| + d(A')) \tag{3.11}$$

(with, say, $c = 1/(4n_0)$). To bound $I'$, we use the inductive assumption (2.17) and the $L^1$ bounds for $\tilde{T}$ in Lemma 2 (stated at the end of this section). The latter imply that the $u$ and the $v$ integrals are bounded by $C^{|A'|}$. Inequality (3.11) implies

$$\delta^{d(A)} \le \delta^{cL^2 d(A')}e^{-c'L^2|A'|}.$$

The sum over $\mathbf{n}$ is bounded by $L^{2|A'|}$; thus, we obtain, for $L$ large enough, since, see (2.16), $\delta^{cL^2} \le \delta'$, and $|A| = |A'|$,

$$I' \le (CL^{2+d}e^{-c'L^2}\epsilon)^{|A'|}\delta'^{d(A')} \le \frac{1}{2}\epsilon^{|A'|}\delta'^{d(A')}. \tag{3.12}$$

Let next $d(A') = 1$. This means that

$$b'_{A'}(z') = \prod_{i=1}^{k} b'_{t'}(u'_i, v'_i)\prod_{j=1}^{l} b'_{t'+1}(u'_{k+j}, v'_{k+j}), \tag{3.13}$$

where both products have at most $n_0$ elements. Here, we need to use the property $\int dvb(t,u,v) = 0$ to get the result. It allows us replace

$$T^{L^2-n-1}(Lv' - v) \to T^{L^2-n-1}(Lv' - v) - T^{L^2-n-1}(Lv' - u) \tag{3.14}$$

in $N_{\mathbf{n}}$ for the terms with $L^2 - n - 1 > 0$. Let us assume, for the moment, that all $n(t')$ and all $L^2 - n(t') - 1$ are different from zero in (3.4). Since $\tau_A(z) \ge \sum_i |u_i - v_i| + \sum_j |u_{k+j} - v_{k+j}|$ we have

$$e^{\lambda\tau_A(z)/L} \le e^{\lambda\tau_A(z)}e^{-\sum_j|u_{k+j}-v_{k+j}|}. \tag{3.15}$$

Then, the right-hand side of (3.8) is replaced by

$$I' \le L^{d|A'|} \sum_{\mathbf{n}} \int du dv \tilde{M}_{\mathbf{n}}(u', u) \tilde{Q}_{\mathbf{n}}(v', v, u) e^{\lambda \tau_A(z)/L} |\langle b_A(z) \rangle^c| \tag{3.16}$$

where

$$\tilde{Q}_{\mathbf{n}}(v', v, u) = \prod_{t'} S^{L^2 - n(t') - 1}_{v_t - u_t}(L v'_{t'} - v_t) \quad \text{and} \tag{3.17}$$

$$S^n_u(v) = |T^n(v) - T^n(v+u)| e^{2\frac{\lambda}{L}|v| - c|u|/2}, \tag{3.18}$$

where $c > 0$ will be chosen below small enough (see (3.35)). Here, because of (3.15), we only need $c/2 \le 1$.

Next, write $\tilde{M}_{\mathbf{n}} = \tilde{M}^1_{\mathbf{n}} \tilde{M}^2_{\mathbf{n}}$ corresponding to the two products in (3.13) and similarly for $\tilde{Q}_{\mathbf{n}}$. Using the pointwise bounds of Lemma 2, we bound

$$\tilde{M}^1_{\mathbf{n}} \tilde{Q}^2_{\mathbf{n}} \le C^{|A'|} \prod_i (1 + n_i)^{-d/2} \prod_j (1 + L^2 - n_{k+j})^{-(d+1)/2}. \tag{3.19}$$

Using the $L^1$ bounds,

$$\int du dv \tilde{M}^2_{\mathbf{n}} \tilde{Q}^1_{\mathbf{n}} \le C^{|A'|}. \tag{3.20}$$

Thus

$$I' \le L^{d|A'|} (C\epsilon)^{|A'|} \sum_{\mathbf{n}} \delta^{d(A)} \prod_i (1 + n_i)^{-d/2} \prod_j (1 + L^2 - n_{k+j})^{-(d+1)/2}. \tag{3.21}$$

The sum can be controlled by the factor $\delta^{d(A)}$ since, for a given $d(A)$, there are at most $d(A)^{|A|} \le d(A)^{2n_0}$ terms, since $|A| = |A'| \le 2n_0$. If $d(A) \ge L^2/2$, we bound the products in (3.21) by 1; so, (3.21) is bounded by $(CL^d\epsilon)^{|A'|}(C\delta)^{L^2/2}$, and we use $|A'| \le 2n_0$, and $CL^{2dn_0}\delta^{L^2/2} \le \delta'$, $d(A') = 1$, to obtain (3.5).

If $d(A) < L^2/2$, since $d(A) \ge \max_i (L^2 - n_i) + \max_j n_{k+j}$, we have in the sum, $n_i \ge L^2/2$, $L^2 - n_{k+j} \ge L^2/2$, and the sum can still be controlled by the factor $\delta^{d(A)}$. So, the sum is bounded by $(C\delta)L^{-d|A'|}L^{-l}$ (since $d(A) \ge 1$) and

$$I' \le (C\epsilon)^{|A'|} L^{-1} (C\delta)^{d(A')} \le \frac{1}{2} \epsilon^{|A'|} \delta'^{d(A')}, \tag{3.22}$$

using $C^{|A'|} L^{-\frac{1}{2}} \le \frac{1}{2}$ (since $|A'| \le 2n_0$), (2.16) and $d(A') = 1$.

If some $n(t')$ or $L^2 - n(t') - 1$ equal zero in (3.4), then, since $d(A) \ge \max_i (L^2 - n_i) + \max_j n_{k+j}$, we have $d(A) \ge L^2 - 1$. We use $T^0(x) = \delta(x) \le L^{nd}$ (on the $n$th scale). There are at most $2n_0$ such factors, and, by the definition (2.16) of $\delta_n$, $(CL)^{cn} \delta_n^{L^2-1} \le \frac{1}{2}\delta'$ (with $c = 2n_0 d$), for $L$ large enough.

This, (3.22) and (3.12) prove (3.5) for $d(A') > 0$.

Let finally $d(A') = 0$. This means that we need to study

$$G_n(u, v) = \left\langle \prod_{i=1}^{k} b_{nt}(u_i, v_i) \right\rangle^c \tag{3.23}$$

where the product has at most $n_0$ elements, and $k \ge 2$.

Define the linear map

$$\mathcal{L}_n G(u', v') = L^{kd} \sum_{t_1+t_2=L^2-1} (T_n^{\otimes k})^{t_1} G (T_n^{\otimes k})^{t_2} (Lu', Lv'). \tag{3.24}$$

$\mathcal{L}_n$ is the part of the linearized RG which involves $G_n$. The full RG is given by

$$G_{n+1} = \mathcal{L}_n G_n + g_n + h_n \tag{3.25}$$

where $g_n$ collects the terms in the linear RG (3.3) with $d(A) > 0$ and $h_n$ the nonlinear contributions in (3.1). The statement (2.17) of Proposition 1, for $d(A') = 0$, amounts to showing

$$\|G_n\| \le C\epsilon^k \tag{3.26}$$

uniformly in $n$. Proceeding as above, we have

$$\|g_n\| \le C\epsilon^k \delta_n \tag{3.27}$$

and in Sect. 4 we will prove that

$$\|h_n\| \le C\epsilon^k \delta_n. \tag{3.28}$$

Thus, to prove (3.26) we need to control $\mathcal{L}_n$. Note that $\mathcal{L}_n$ is the derivative of the map

$$G \to L^{kd} G^{L^2}(L\cdot) \tag{3.29}$$

computed at $G = T_n^{\otimes k}$. Let $\mathcal{L}^*$ similarly be computed with $T_D^*$. The bound (2.20) in Proposition 2 and (3.7) imply

$$\|\mathcal{L}^* - \mathcal{L}_n\| \le C\delta_n, \tag{3.30}$$

Hence, to prove (3.26) it suffices to bound $\|\mathcal{L}^{*n}\|$ uniformly in $n$, as in (3.32) below. Indeed, if this is the case, then, (3.30) implies a uniform bound on $\|\prod_{\ell=k}^n \mathcal{L}_\ell\|$ in $k, n$, by $C' \prod_i (1 + C\delta_i))$, which is finite by (2.16). Then, we get from (3.27), (3.28), by iterating (3.25),

$$\|G_n\| \le C \sum_{j=0}^{n} \left\| \prod_{\ell=j+1}^{n} \mathcal{L}_\ell \right\| \epsilon^k \delta_j.$$

Since $\sum_j \delta_j < \infty$ by (2.16), this implies (3.26).

Actually, $\|\mathcal{L}^{*n}\|$ is not uniformly bounded, but, instead, we have the following lemma, which allows us to conclude the proof of (3.26), since, by (2.10), (3.31) holds for $b_{nt}(u_i, v_i)$.

**Lemma 1** *Let $G$ satisfy*

$$\int dv_i G(u, v) = 0 \tag{3.31}$$

*for $i = 1, \dots, k$. Then, $\exists C < \infty$, such that*

$$\|\mathcal{L}^{*n} G\| \le C\|G\| \tag{3.32}$$

*uniformly in $n$. Moreover,*

$$\sup_u \int dv e^{\frac{1}{2}\lambda\tau_A(z)} |\mathcal{L}^{*n} G(u,v)| \le C L^{-n} \log L^n \|G\|_1 \le \frac{1}{2}\delta_n \|G\|_1, \tag{3.33}$$

*where $\|G\|_1$ denotes the norm in* (2.18).

Using this lemma, we prove (3.24) following the proof of (3.26), using $\|G\|_1 \le C\|G\|$, (3.27), (3.28), and (3.33), which can be written as $\|\mathcal{L}^{*n} G\|_1 \le \frac{1}{2}\delta_n \|G\|_1$.

*Proof of Lemma 1* Denote explicitly the $L$ dependence of $\mathcal{L}^*_L$. We have $\mathcal{L}^{*n}_L = \mathcal{L}^*_{L^n}$, because the map (3.29) applied $n$ times is the same as (3.29) applied once with $L$ replaced by $L^n$. Hence we need to study the large $L$ behavior of $\mathcal{L}^*_L$. The summand in (3.24) is explicitly given by (dropping the star)

$$L^{kd} \int du dv \prod_{i=1}^k T^{t_1}(Lu'_i - u_i) T^{t_2}(Lv'_i - v_i) G(u,v). \tag{3.34}$$

Using (3.31) we may again subtract $T^{t_2}(Lv'_i - u_i)$ from each $T^{t_2}(Lv'_i - v_i)$ when $t_2 > 0$, which means that we replace $T^{t_2}(Lv'_i - v_i)$ in (3.34) by $T^{t_2}(Lv'_i - v_i) - T^{t_2}(Lv'_i - u_i)$. Recalling (1.10), we write, instead of (3.15),

$$\frac{\lambda\tau_A(z)}{L} \le \lambda\tau_A(z) - \sum_i |u_i - v_i| - \frac{\lambda\tau(S(z))}{2}.$$

Since $\tau(S(z))$ is the length of a graph on $S(z)$, $\tau(S(z)) \ge |u_i - u_{i+1}|$ for all $i$, and thus $\tau(S(z)) \ge \sum_i |u_i - u_{i+1}|/(k+l)$; so, combining the argument here with (3.7), we get:

$$\lambda\tau_{A'}(z') \le \lambda\tau_A(z) - c\left(\sum_i |u_i - v_i| + |u_i - u_{i+1}|\right) + \frac{2\lambda}{L}\sum_{t'}(|Lu'_{t'} - u_t| + |Lv'_{t'} - v_t|), \tag{3.35}$$

where, since $k + l \le 2n_0$, $c$ depends only on $n_0$ and $\lambda$.

Then the supremum over $z'$ of (3.34) multiplied by $e^{\lambda\tau_{A'}(z')}$ is bounded, using Lemma 2 (where we use bounds on $S^{t_2}_{v_i - u_i}(Lv'_i - v_i)$, using definition (3.18)), by

$$\begin{aligned} &L^{kd}(1+t_1)^{-kd/2}(1+t_2)^{-k(d+1)/2} \\ &\quad \times \int du dv e^{-c'(|Lu'_1 - u_1|/\sqrt{t_1} + |Lv'_1 - v_1|/\sqrt{t_2})} e^{-c\sum_i(|u_i - v_i| + |u_i - u_{i+1}|)/2} \|G\|. \end{aligned} \tag{3.36}$$

The factor $e^{-c\sum_i(|u_i - v_i| + |u_i - u_{i+1}|)/2}$ allows us to integrate over all the variables (of which there are most $2n_0$), except one, say $u_1$. And, using $|Lv'_1 - v_1| + |v_1 - u_1| \ge |Lv'_1 - u_1|$, for the integration over $u_1$, the integral is bounded by:

$$C \int du_1 e^{-c''(|Lu'_1 - u_1|/\sqrt{t_1} + |Lv'_1 - u_1|/\sqrt{t_2})} \tag{3.37}$$

which in turn is bounded by $C(1+t_i)^{d/2}$ where we use $i = 1$ if $t_1 < L^2/2$ and $i = 2$ if $t_1 \ge L^2/2$. Let us divide the sum over $t_1$ of (3.36) into one with $t_1 < L^2/2$ and another with

$t_1 \geq L^2/2$. In the first sum, we use $t_2 \geq L^2/2$ to control the $L^{kd}$ factor, and in the second sum, we use $t_1 \geq L^2/2$. The result is that the sum is bounded by:

$$C \sum_{0 \leq t \leq L^2/2} (L^{-k}(1+t)^{-(k-1)d/2} + (1+t)^{-((k-1)d/2+k/2)})\|G\|. \tag{3.38}$$

This is uniformly bounded in $L$ for all $d \geq 1$ and $k \geq 2$. The first claim follows.

For the second one, we integrate (3.34) also over $v'$, which absorbs the factor $L^{kd}$ through the change of variables $v' \to Lv'$. Using Lemma 2 for the $L^1$ norm of $S^{t_2}_{v_i-u_i}(Lv'_i - v_i)$ integrated over $Lv'_i$ and (3.35), we get that (3.34), multiplied by $e^{\frac{1}{2}\lambda\tau_{A'}(z')}$, and integrated over $v'$, is bounded by

$$C(1+t_2)^{-k/2}\|G\|_1 \int du \prod_{i=1}^{k} \tilde{T}^{t_1}(Lu'_i - u_i)e^{-c\sum|u_i-u_{i+1}|}. \tag{3.39}$$

Use Lemma 2 with $L^\infty$ norm for $k-1$ $\tilde{T}$'s and $L^1$ norm for one $T$ to bound the integral by $C(1+t_1)^{-(k-1)d/2}$. Altogether we end up with a bound for the LHS of (3.33) (with $L^n$ replaced by $L$)

$$C \sum_{0 \leq t_1 \leq L^2} ((1+t_1)^{-(k-1)d/2}(1+t_2)^{-k/2})\|G\|_1 \leq CL^{-1}\log L\|G\|_1. \tag{3.40}$$

□

The proof of the following lemma if deferred to Sect. 4.

**Lemma 2** *Let $T = T_n$. There exists $C < \infty$, $c > 0$ such that, for $L > L(\lambda)$, we have, using definitions* (3.9), (3.18),

$$\tilde{T}^m(u) \leq Cm^{-\frac{d}{2}}e^{-c|u|/\sqrt{m}}, \qquad \|\tilde{T}^m\|_1 \leq C,$$

*where $c$ can be chosen equal to 1 for $n \geq 1$, and*

$$\|S^m_u\|_\infty \leq Cm^{-\frac{d+1}{2}}, \qquad \|S^m_u\|_1 \leq Cm^{-\frac{1}{2}}$$

*for all $m \in [1, L^2]$. We also have $\|\tilde{T}^0\|_1 \leq C$.*

## 4 Proof of Proposition 1

As before, we drop the index $n$ and denote $n+1$ by prime. Using the notation introduced in Sect. 1 we may expand the product over $t$, and write (3.1) as

$$b'(t', z') = L^d \sum_A \int dz K_A(z', z)\big(b_A(z) - \langle b_A(z)\rangle\big), \tag{4.1}$$

where the sum runs over subsets of $I_{t'}$,

$$K_A(z', z) = \prod_{i=0}^{l} T^{t_{i+1}-t_i-1}(v_i - u_{i+1}), \tag{4.2}$$

with $T^0(u) = \delta(u)$. $K$ depends on $z'$ through $v_0 = Lu'$, $u_{l+1} = Lv'$. We have $|A| = l$.

Equation (4.1) leads to the following recursion relation for the cumulants:

**Lemma 3** *Let $A'$ be of the form (1.8) i.e. $A' = \coprod A'_i$. Then*

$$\langle b'_{A'}(z')\rangle^c = L^{d|A'|} \sum_{\mathcal{A}} \sum_{\Pi\in\mathcal{P}^c_{A'}(A)} \int dz \prod_{t'\in A'} K_{A_{t'}}(z'_{t'}, z) \prod_{B\in\Pi} \langle b_B(z)\rangle^c \tag{4.3}$$

*where $\mathcal{A} = \{A_{t'}\}_{t'\in A'}$ is a family of sets $A_{t'} \subset I_{t'}$ and $\mathcal{P}^c_{A'}(A)$ is the set of partitions of $A = \coprod A_{t'}$ that "connect" $A'$ i.e. so that the following graph is connected: its set of vertices is $A'$ and its set of edges are the pairs $\{t', t''\}$ such that, for some $B \in \Pi$, both $B \cap A_{t'}$ and $B \cap A_{t''}$ are nonempty.*

Now, the iteration of (2.17) follows the lines of Sect. 3, starting from (4.3) instead of (3.3). We need the analogues of (3.6) and (3.7). To state them we need some notation.

First, write, for $t' \in A'$, $A_{t'} = \{t_{t'i} \mid i = 1, \dots, |A_{t'}|\}$. Let $z_{t_{t'i}} = (u_{t'i}, v_{t'i})$ and $v_{t'0} = Lu'_{t'}$, $u_{t'|A_{t'}|+1} = Lv'_{t'}$.

It will also be important to single out the linear term in (4.1). For this, let $S' \subset A'$ consist of those $t'$ for which $A_{t'}$ consists of a single time, call it $t_{t'}$, and let $S = \coprod_{t'\in S'} t_{t'}$. Note that Sect. 3 dealt with the case where $S' = A'$.

**Lemma 4** *Let $A' \setminus S' \neq \emptyset$. For any value of $z$ in (4.3),*

$$\tau_{A'}(z') \le \frac{1}{L}\left(\sum_{B\in\Pi} \tau_B(z) + 2\sum_{t'\in A'} \sum_{i=0}^{|A_{t'}|} |v_{t'i} - u_{t'i+1}|\right), \tag{4.4}$$

$$d(S) \le \sum_{B\in\Pi} d(B) + 2L^4|A' \setminus S'|, \tag{4.5}$$

$$d(A') \le \min\left\{\frac{2}{L^2}\sum_{B\in\Pi} d(B) + 2L^2|A' \setminus S'|, \sum_{B\in\Pi} d(B)\right\}. \tag{4.6}$$

Let $\tilde{K}_{A_{t'}}(z', z)$ be given by (4.2) with $T^t$ replaced by $\tilde{T}^t$ (see (3.9)). Then, inserting (4.4) into (4.3), we get

$$\|\langle b'_{A'}\rangle^c - \langle(\mathcal{L}b)_{A'}\rangle^c\| \le L^{d|A'|} \sum_{S'\neq A'} \sum_{S,\mathcal{B}} \sum_{\Pi\in\mathcal{P}^c_{A'}(A)} I(S', S, \mathcal{B}, \Pi) \tag{4.7}$$

where the $\mathcal{B}$ sum is over $A_{t'}$ with $t' \in A' \setminus S'$ i.e. such that $|A_{t'}| > 1$. We introduced also

$$I(S', S, \mathcal{B}, \Pi) = \sup_{z'} \int dz \prod_{t'\in A'} \tilde{K}_{A_{t'}}(z', z) \prod_{B\in\Pi} e^{\frac{\lambda}{L}\tau_B(z)} |\langle b_B(z)\rangle^c|. \tag{4.8}$$

To bound (4.8), use again $\sum_{t\in B} |u_t - v_t| \le \tau_B(z)$, see (1.10), which allows us to replace each $\tilde{K}_{A_{t'}}$ by

$$\tilde{K}_{A_{t'}}(z', z) \prod_{t\in A_{t'}} e^{-c|u_t - v_t|} \tag{4.9}$$

at the cost of replacing $\frac{\lambda}{L}$ in the exponent in (4.8) by $\lambda$. The integral of (4.9) over $u$ and $v$ is bounded by a convolution of $2|A_{t'}|$ $L^1$ functions whose $L^1$-norm is $\mathcal{O}(1)$, by Lemma 2.

Thus, since $|A|=\sum_{t'}|A_{t'}|$,

$$I(S',S,\mathcal{B},\Pi)\le C^{|A|}\prod_{B\in\Pi}\|\langle b_B\rangle^c\|. \tag{4.10}$$

From our inductive assumption (2.17), we get

$$\prod_{B\in\Pi}\|\langle b_B\rangle^c\|\le (C\epsilon)^{|A|}\delta^{\sum d(B)}. \tag{4.11}$$

Recall that $\delta=L^{-\frac{1}{2}n}e^{-\lambda}$. Let first $n>0$. Taking convex combination of the bounds in (4.6), we have

$$d(A')\le(1-x+2x/L^2)\sum d(B)+2xL^2|A'\setminus S'|,$$

and choosing $1-x+2x/L^2=(n-\frac{1}{2})/(n+1)$,

$$n\sum d(B)\ge(n+1)d(A')+\frac{1}{2}\sum d(B)-cL^2|A'\setminus S'|,$$

where $c$ is independent of $n$, since $x=\mathcal{O}(n^{-1})$, as $n\to\infty$. So,

$$\prod_{B\in\Pi}\|\langle b_B\rangle^c\|\le C(L)^{|A'\setminus S'|}(C\epsilon)^{|A|}\delta'^{d(A')}\eta^{3\sum d(B)} \tag{4.12}$$

where $\eta^3=L^{-1/4}$. For $n=0$ take $x=1$ and (4.12) follows with $\eta^3=e^{-\lambda/4}$, using $e^{-\lambda L^2/4}\le\delta'=\delta_1$.

Let us insert (4.12), (4.10) into (4.7), and then turn to the four sums in (4.7). To control them, we use the three factors $\eta^{\sum d(B)}$ in (4.12). For the sum over partitions, we use the simple bound

$$\sum_{\Pi\in\mathcal{P}(A)}\prod_{B\in\Pi}\eta^{d(B)}\le C^{|A|}, \tag{4.13}$$

which holds for $\eta$ small enough, since the left-hand side of (4.13) is bounded by

$$\prod_{t\in A}\Bigg(\sum_{t\in B\subset A}\eta^{d(B)}\Bigg)\le C^{|A|}. \tag{4.14}$$

Consider next the $\mathcal{B}$ sum. Since each $A_{t'}\in\mathcal{B}$ is a subset of size at least two of a set of $L^2$ points we have (recall that $|A|=|S'|+\sum_{t'\in A'\setminus S'}|A_{t'}|$)

$$\sum_{\mathcal{B}}(C\epsilon)^{|A|}\le(C\epsilon)^{|S'|}(C(L)\epsilon^2)^{|A'|-|S'|}. \tag{4.15}$$

Finally, for the sum over $S$, use (4.5) to write $\eta^{\sum d(B)}\le(C(L))^{|A'|-|S'|}\eta^{d(S)}$. Then, write the elements of $S$, $\{t_{t'}\}$, $t'\in S'$ as $t_1\le t_2\le\cdots\le t_{|S'|}$ so that $d(S)=|t_{|S'|}-t_1|$. Then

$$\sum_S\eta^{d(S)}\le\sum_{t_1\le\cdots\le t_{|S'|}}\eta^{|t_{|S'|}-t_1|}\le L^2C^{|S'|}\le C(L)^{|A'\setminus S'|}C^{|S'|}, \tag{4.16}$$

since at most $n_0$ times may coincide. The $L^2$ factor comes from the sum over $t_1$ and the last inequality uses $A'\setminus S'\neq\emptyset$.

We need also to bound the factor $L^{d|A'|}$ in (4.7). We write $|A'| = |A' \setminus S'| + |S'|$. If $d(S') \le 1$, we have $|S'| \le 2n_0$, and we can bound $L^{d|S'|}$ by $C(L)^{|A'\setminus S'|}$, since $|A' \setminus S'| \neq 0$. If $d(S') > 1$, we use (3.10) to bound $|S'| \le n_0(d(S')+1) \le cd(S)/L^2$, and use (4.5) for $d(S)$. Altogether, this gives, using the last factor $\eta^{\sum d(B)}$ in (4.12),

$$
\begin{aligned}
L^{(d+1)|A'|}\eta^{\sum d(B)} &\le C(L)^{|A'\setminus S'|} L^{(d+1)|S'|}\eta^{d(S)} \\
&\le C(L)^{|A'\setminus S'|} L^{(d+1)|S'|}\eta^{c' L^2 |S'|} \le C(L)^{|A'\setminus S'|}. \qquad (4.17)
\end{aligned}
$$

So, we get:

$$
L^{d|A'|}\eta^{\sum d(B)} \le C(L)^{|A'\setminus S'|} L^{-|A'|} \qquad (4.18)
$$

where the factor $L^{-|A'|}$ will be used now.

Combining (4.10), (4.12), (4.13), (4.15), (4.16) and (4.18), we get, for $\epsilon$ small,

$$
(4.7) \le \delta'^{d(A')} L^{-|A'|} \sum_{S' \neq A'} (C\epsilon)^{|S'|} (C(L)\epsilon^2)^{|A'|-|S'|} \le \tfrac{1}{2}\delta'^{d(A')}\epsilon^{|A'|}, \qquad (4.19)
$$

since the sum equals $(C\epsilon + C(L)\epsilon^2)^{|A'|} - (C\epsilon)^{|A'|} \le (C'\epsilon)^{|A'|}$ and we use $L^{-|A'|}$ to control $C'^{|A'|}$. Combining (4.7), (4.19), and (3.5), (2.17) is proven for $d(A') > 0$.

For $d(A') = 0$, we obtain a bound similar to (4.19) on $h_{n+1}$ defined in (3.25), with $\delta = \delta_n$ instead of $\delta'^{d(A')}$, since all the terms in (4.7) have at least one power of $\delta$. Using part of the factor $L^{-|A'|}$ in (4.19), we can replace $\delta$ by $\delta'$ which proves the bound (3.28) for $h_{n+1}$. Combining (3.28) with (3.27) and (3.32) finishes the proof of (3.26), i.e. of (2.17) for $d(A') = 0$, while using (3.28) with (3.27) and (3.33) finishes the proof of (2.18). □

We are left with the proofs of the lemmas.

*Proof of Lemma 3* Using (4.1) for $b'$, we get

$$
\langle b'_{A'}(z')\rangle = L^{d|A'|} \sum_{\{A_{t'}\}_{t'\in A'}} \int dz \prod_{t'\in A'} K_{A_{t'}}(z'_{t'}, z) \Big\langle \prod_{t'} \big(b_{A_{t'}}(z) - \langle b_{A_{t'}}(z)\rangle\big) \Big\rangle \qquad (4.20)
$$

where $|A'| = \sum_i |A'_i|$ (note that here, up to $n_0$ of the times $t'$ may coincide). To get connected correlations, use first the inverse of (1.9):

$$
\langle b_A(z)\rangle = \sum_{\Pi \in \mathcal{P}(A)} \prod_{B\in\Pi} \langle b_B(z)\rangle^c \qquad (4.21)
$$

to obtain a recursion formula for

$$
\Big\langle \prod_{t'} (b_{A_{t'}}(z) - \langle b_{A_{t'}}(z)\rangle) \Big\rangle = \sum_{\Pi\in\mathcal{P}_{A'}(\coprod_{t'} A_{t'})} \prod_{B\in\Pi} \langle b_B(z)\rangle^c \qquad (4.22)
$$

where $\mathcal{P}_{A'}$ is the set of partitions such that no $B \in \Pi$ is a subset of $A_{t'}$ for some $t'$.

Inserting (4.22) into (4.20) and denoting $A = \coprod_{t'} A_{t'}$, we get:

$$
\langle b'_{A'}(z')\rangle = L^{d|A'|} \sum_{\{A_{t'}\}} \sum_{\Pi\in\mathcal{P}_{A'}(A)} \int dz \prod_{t'\in A'} K_{A_{t'}}(z'_{t'}, z) \prod_{B\in\Pi} \langle b_B(z)\rangle^c. \qquad (4.23)
$$

To prove (4.3), consider a $\Pi$ in (4.23) and associate to it a graph on $A'$ by connecting pairs $\{t', t''\}$ such that, for some $B \in \Pi$, both $B \cap A_{t'}$ and $B \cap A_{t''}$ are nonempty. Decompose that graph into connected components, $B'_i$, and write $A' = \bigcup_i B'_i$. This defines a partition of $A'$. Now, observe that the sum in (4.23) factorizes over those connected components:

$$\langle b'_{A'}(z')\rangle = \sum_{\Pi\in\mathcal{P}(A')}\prod_{B'\in\Pi}\left(L^{d|B'|}\sum_{\{A_{t'}\}_{t'\in B'}}\sum_{\Pi\in\mathcal{P}^c_{B'}(A)}\int dz\prod_{t'\in B'}K_{A_{t'}}(z'_{t'},z)\prod_{B\in\Pi}\langle b_B(z)\rangle^c\right)$$

where, for each factor in the product over $B'$, we write $A = \coprod_{t'\in B'} A_{t'}$. Now, write (4.21) with primes and observe that (4.21) uniquely determines the connected correlation function (because it is the inverse of (1.9)) to obtain (4.3). □

*Proof of Lemma 4* Let $G_B$ be a connected graph whose set of vertices include $z_t$ for $t\in B$ and whose length equals $\tau(S(z_B))$ (we denote the restriction of $z$ to $B$ by $z_B$). Let $E$ be the graph obtained by joining to the union of the $G_B$ the lines with endpoints $v_{t'i}$ and $u_{t'i+1}$ for each $i=0,\dots,|A_{t'}|$, $t'\in A'$. We claim that $E$ is connected.

To see this observe first that any two points within the same $S(z_{A_{t'}})$ are connected by a path in $E$, since each $u_{t'i}$ is connected to $v_{t'i}$ (because they belong to the same $S(z_B)$), and each $v_{t'i}$ is connected to $u_{t'i+1}$ by the additional lines.

Next, consider $w,\tilde w\in\bigcup_{t'\in A'}S(z_{A_{t'}})$. Since each $\Pi$ in (4.3) connects $A'$, there exists a sequence $A_{t'_1},\dots,A_{t'_\ell}$ with $w\in S(z_{A_{t'_1}})$, $\tilde w\in S(z_{A_{t'_\ell}})$, and a sequence $(B_i)_{i=1}^{\ell}$ such that $B_i\cap A_{t'_i}\neq\emptyset$, $B_i\cap A_{t'_{i+1}}\neq\emptyset$, $i=1,\dots,\ell-1$. So, we have $S(z_{B_i})\cap S(z_{A_{t'_{i+1}}})\neq\emptyset$, $S(z_{B_{i+1}})\cap S(z_{A_{t'_{i+1}}})\neq\emptyset$, $i=1,\dots,\ell-1$. Since the graph $E$ connects each of the sets $S(z_{A_{t'}})$ and since there are points in $S(z_{B_i})$ and $S(z_{B_{i+1}})$ that belong to the same $S(z_{A_{t'}})$, and thus, by the previous observation, are connected by a path in $E$, we see that there exists a connected path in $E$ joining $w$ and $\tilde w$.

Since the set of vertices of $E$ contains $S(Lz')$, and $E$ is connected, its length is larger than $\tau(S(Lz')) = L\tau(S(z'))$. By construction, the length of $E$ equals $\sum_{B\in\Pi}\tau(S(z_B)) + \sum_{t'\in A'}\sum_{i=0}^{|A_{t'}|}|v_{t'i}-u_{t'i+1}|$ so we get:

$$\tau(S(z'))\le\frac{1}{L}\left(\sum_{B\in\Pi}\tau(S(z_B))+\sum_{t'\in A'}\sum_{i=0}^{|A_{t'}|}|v_{t'i}-u_{t'i+1}|\right). \tag{4.24}$$

Since also

$$|Lu'_{t'}-Lv'_{t'}|\le\sum_{i=0}^{|A_{t'}|}(|u_{t'i}-v_{t'i}|+|v_{t'i}-u_{t'i+1}|)$$

the claim (4.4) follows from the definition (1.10).

Next we prove (4.5). Let $\Pi_S\subset\Pi$ be the set of $B\in\Pi$ that contain elements of $S$. Note that each $B\in\Pi_S$ has to contain elements of $A\setminus S$, since $\Pi$ connects $A'$, unless $S=A$, which is not possible since $A'\setminus S'\neq\emptyset$ by assumption.

Let then $A\setminus S\neq\emptyset$, so that we can assume that each $B\in\Pi_S$ contains elements of $A\setminus S$. For $B\in\Pi$, let $I_B=[s_B,t_B]$ where $s_B$, (resp. $t_B$) is the minimal (maximal) time in $B$. Hence $d(B)=t_B-s_B$. Let $\sigma=\min_{B\in\Pi_S}s_B$, $\tau=\max_{B\in\Pi_S}t_B$. Let $\bar I$ denote the smallest interval of $L^2\mathbb{N}$ containing $I$, an interval in $\mathbb{N}$. Then the sets $B\in\Pi\setminus\Pi_S$ connect the $\bar I_B$'s in $\Pi_S$. As a consequence

$$\left|[\sigma,\tau]\setminus\bigcup_{B\in\Pi_S}\bar I_B\right|\le\sum_{B\in\Pi\setminus\Pi_S}d(B).$$

Also, since $|\bar{I}_B| \le d(B) + 2L^2$, $|\bigcup_{B\in\Pi_S} \bar{I}_B| \le \sum_{B\in\Pi_S} d(B) + 2L^2|\Pi_S|$. Since each $B \in \Pi_S$ contains elements in $A \setminus S$, $|\Pi_S| \le |A \setminus S| \le L^2|A' \setminus S'|$ and thus

$$d(S) \le |\tau - \sigma| \le \sum_{B\in\Pi} d(B) + 2L^4|A' \setminus S'|$$

and (4.5) is proven.

Finally we prove (4.6). Since $S' \ne A'$, then, as before, each $B \in \Pi$ contains elements in $A \setminus S$ and hence $|\Pi| \le L^2|A' \setminus S'|$. For each $B$ we have $d(B) \ge |\bar{I}_B| - 2L^2$. Also, $d(L^2A') \le \sum_{B\in\Pi} |\bar{I}_B|$. These imply

$$L^2 d(A') = d(L^2A') \le \sum_{B\in\Pi} d(B) + 2L^4|A' \setminus S'|$$

which is our claim, since the bound $d(A') \le \sum_{B\in\Pi} d(B)$ holds trivially. □

To prove Lemma 2, we need a lemma on $T_n$, which will be proven in the next section, since it will be also the basis of the proof of Proposition 2. Note that the Fourier transform $\hat{T}_n$ is defined on the torus $\mathbb{T}_n = (L^n\mathbb{T})^d$. The properties of $\hat{T}_n$ are summarized in the following lemma (where the domain of analyticity and the bounds are sufficient for our proofs but not optimal).

**Lemma 5** *Let $\hat{T}$ be as in* A.2. *Then there exists $r, c > 0$ such that for all $n \ge 0$, $\hat{\mathcal{T}}_n$, defined in* (2.12), *is analytic in $|\Im k| < r^2 L^{\frac{n}{4}}$ and, for such $k$,*

$$\hat{\mathcal{T}}_n(k) = (1 + \mathcal{O}(L^{-2n}|k|^4))e^{-\frac{D_0}{2d}k^2} \tag{4.25}$$

*if $|k| \le rL^{\frac{n}{4}}$ and*

$$|\hat{\mathcal{T}}_n(k)| \le e^{-cL^{\frac{n}{2}}} \tag{4.26}$$

*otherwise.*

*Moreover, under the assumptions of Theorem 1, $T_n$ can be expressed as*

$$\hat{T}_n(k) = \hat{\mathcal{T}}_n(\rho_n k) + \hat{t}_n(k) \tag{4.27}$$

*where*

$$|\rho_n - 1| \le C\epsilon^2, \tag{4.28}$$

*for $C < \infty$, with $\rho_0 = 1$. We have $t_0 = 0$,*

$$|\hat{t}_n(k)| \le \epsilon\delta_n|k|^4, \quad \text{for } |k| \le 1, \tag{4.29}$$

*and*

$$|t_n(x)| \le \epsilon\delta_n e^{-2|x|}. \tag{4.30}$$

*Proof of Lemma 2* Let $n \ge 0$ and $\mathcal{T} := \mathcal{T}_n$. Let $m \in [1, L^2]$ and consider $\mathcal{T}^m(x) = \int dk\hat{\mathcal{T}}(k)^m e^{ikx}$. Shift the $k$ integration by $ip$ (to be precise, by $\pm ip$ for each coordinate $j$, depending on the sign of $x_j$), with $p^2 = a^2/m < r^4$. Then we get, using (4.25) and (4.26),

$$\mathcal{T}^m(x) \le Ce^{-a|x|/\sqrt{m}}(e^{ca^2}m^{-\frac{d}{2}} + e^{-cm}),$$

where we divided the integral into $|k|\le 1$ and $|k|>1$, and used $|e^{-\frac{D_0}{2d}k^2}|\le e^{-c|k|^2}$ for $|k|>1$, $|\Im k|<r^2$. Clearly, for $L$ large, we may, for all $m\in[1,L^2]$, choose $a=a(m)=\min(r^2\sqrt{m}/2,3\lambda)$, so that, for all $1\le m\le L^2$ and $L$ large enough, $a/\sqrt{m}-2\lambda/L\ge c>0$ and $a^2/m<r^4$. Hence,

$$\tilde{\mathcal{T}}^m(x)\le Cm^{-\frac{d}{2}}e^{-c|x|/\sqrt{m}} \tag{4.31}$$

where $\tilde{\mathcal{T}}$ is defined as in (3.9) and $C$ and $c$ depend only on $T$ and $\lambda$.

From (4.30) we obtain that $\hat{t}(k)$ is analytic for $|\Im k|<2$ and is bounded by $C\epsilon\delta_n$. Since this is less than $m^{-\frac{d}{2}}$ for $m\le L^2$, we can repeat for $\hat{T}(k)$ the argument given for $\hat{\mathcal{T}}(k)$. For $n\ge 1$, the domain of analyticity of $\hat{\mathcal{T}}(k)$ can be taken as large as one wants (by choosing $L$ large) and we can choose $a$ large enough so that we can have $c=1$ in (4.31). Since $c=1$ is $L$ independent, the constants $C$ in (4.31) still depends only on $T$ and $\lambda$. This proves the first two claims of Lemma 2.

For the other two claims, observe that, if we prove them for $|u|=1$, then we can interpolate between $v$ and $v+u$ by steps of size 1 and obtain the result. The exponential factor in (3.18) $e^{-c|u|/2}$ controls both the $e^{2\lambda|u|/L}$ coming from the interpolation (for $L$ large) and the number $|u|$ of interpolation steps. Now, for $|u|=1$, we shift again the integration contour:

$$|\mathcal{T}^m(v+u)-\mathcal{T}^m(v)|\le e^{-p|v|}\int\hat{\mathcal{T}}(k)^m|e^{(p-ik)u}-1|dk$$

(where, with an abuse of notation, $p$ denotes a number in $p|v|$ and a vector in $pu$). We take $p=a/\sqrt{m}$, with $a=a(m)$ as above, Using (4.25) and (4.26) we bound the integral by

$$\int_{|k|\le 1}e^{-cmk^2}|e^{(p-ik)u}-1|dk+Ce^{-cm}.$$

The integral is bounded by $C(p+m^{-1/2})m^{-d/2}$. Altogether we get, since $a\le 3\lambda$,

$$|\mathcal{T}^m(v+u)-\mathcal{T}^m(v)|\le C(p+m^{-1/2})m^{-d/2}e^{-p|v|}\le Cm^{-(d+1)/2}e^{-a|v|/\sqrt{m}},$$

where again $C$ depends only on $T$ and $\lambda$. We then get the estimates for $\tilde{T}$ as above, since $\hat{t}$ gives corrections of order $\epsilon\delta_n$. □

## 5 Proof of Proposition 2

We start with the:

*Proof of Lemma 5* Our assumptions on $T$ imply that for $r$ small enough, $|\hat{T}(k)|\le\rho(r)<1$ for $|\Re k|>r$, $|\Im k|\le r^2$; this implies (4.26) for $|k|>rL^n$, $|\Im k|\le r^2L^{\frac{n}{4}}$. Write (2.13) as $\hat{T}(k)=e^{-\frac{D_0}{2d}k^2}(1+\mathcal{O}(|k|^4))$ for $|k|\le r$. This implies (4.25) for $|k|\le rL^n$, in particular for $|k|\le rL^{\frac{n}{4}}$, $|\Im k|\le r^2L^{\frac{n}{4}}$. Since $|e^{-\frac{D_0}{2d}k^2}|=e^{-\frac{D_0}{2d}((\Re k)^2-(\Im k)^2)}\le e^{-\frac{D_0}{4d}|k|^2}$ for $\frac{1}{2}|k|>|\Im k|$, the claim (4.26) holds also for $rL^{\frac{n}{4}}<|k|\le rL^n$, $|\Im k|\le r^2L^{\frac{n}{4}}$, *if* if $r$ is taken small enough.

For the other statements, write again $T$ for $T_n$ and $T'$ for $T_{n+1}$. We have from (4.1)

$$T'(x)=L^dT^{L^2}(Lx)+\beta(x) \tag{5.1}$$

with

$$\beta(x) = L^d \sum_A \int dz K_A(z', z)\langle b_A(z)\rangle, \tag{5.2}$$

where $z' = (0, x)$ and only $A$'s containing distinct times enter. Note that (5.2) collects the averages that have been subtracted in (4.1) and therefore enter the definition of $T'$. Now use (4.21); since $A$ contains only distinct times, each $B$ in (4.21) has $d(B) > 0$. Using (4.4) and $z' = (0, x)$ to bound $e^{2|x|}$, Proposition 1 and Lemma 2 immediately imply the bound

$$|\beta(x)| \le C(L)\epsilon^2\delta e^{-2|x|} \le \frac{1}{2}\epsilon\delta' e^{-2|x|}, \tag{5.3}$$

where $\epsilon^2$ comes because $|A| \ge 2$ and we use $C(L)\epsilon \le 1$.

In terms of Fourier transform, (5.1) reads:

$$\hat{T}'(k) = \hat{T}(k/L)^{L^2} + \hat{\beta}(k). \tag{5.4}$$

By (5.3), $\hat{\beta}$ is analytic in $|\Im k| \le 3/2$ and bounded there by $C(L)\epsilon^2\delta$. By the isotropy Assumption A.2 in Sect. 1, the Taylor expansion reads

$$\hat{\beta}(k) = \zeta k^2 + \mathcal{O}(|k|^4). \tag{5.5}$$

We used $\hat{\beta}(0) = 0$ which follows from (5.4) and $\hat{T}(0) = 1 = \hat{T}'(0)$. By Cauchy's theorem

$$|\zeta| \le C(L)\epsilon^2\delta, \tag{5.6}$$

and

$$|\hat{\beta}(k) - \zeta k^2| \le C(L)\epsilon^2\delta|k|^4, \tag{5.7}$$

for $|k| \le 1$.

The $\beta$ term will "renormalize" the effective diffusion constant $D = \rho D_0$. We set

$$\rho'^2 = \rho^2 - 2d\zeta D_0^{-1}. \tag{5.8}$$

Relations (5.6) and (2.16), which implies the convergence of $\sum_n \delta_n$, then imply the bound (4.28).

Consider next the first term in (5.4) (recall (4.27)):

$$\hat{T}\left(\frac{k}{L}\right)^{L^2} = \left(\hat{T}\left(\rho\frac{k}{L}\right) + \hat{t}\left(\frac{k}{L}\right)\right)^{L^2} := \hat{T}\left(\rho\frac{k}{L}\right)^{L^2} + \hat{\tau}\left(\frac{k}{L}\right). \tag{5.9}$$

Since $\hat{T}'(k) = \hat{T}'(\rho' k) + t'(k)$ we get from (5.4) and (5.9) that

$$\hat{t}'(k) = \hat{\tau}\left(\frac{k}{L}\right) + \hat{\beta}(k) + \hat{r}(k), \tag{5.10}$$

where

$$\hat{r}(k) := \hat{T}'(\rho k) - \hat{T}'(\rho' k), \tag{5.11}$$

since, by definition (2.11), $\hat{T}'(\rho k) = \hat{T}(\rho \frac{k}{L})^{L^2}$.

We need to show that $t'$ satisfies (4.29) and (4.30) with $\delta'$. Consider (4.29) first. From (5.9), we have

$$\hat{\tau}\left(\frac{k}{L}\right)=\sum_{m=1}^{L^2}\binom{L^2}{m}\hat{t}\left(\frac{k}{L}\right)^m\hat{\mathcal{T}}\left(\rho\frac{k}{L}\right)^{L^2-m}. \tag{5.12}$$

By (4.29), $|\hat{t}(\frac{k}{L})|\le\epsilon\delta L^{-4}|k|^4$ and by Lemma 5, $|\hat{\mathcal{T}}|$ is bounded for $|k|\le 1$. Hence

$$\left|\hat{\tau}\left(\frac{k}{L}\right)\right|\le C\epsilon\delta L^{-4}|k|^4(L^2+C(L)\epsilon\delta)\le\frac{1}{2}\epsilon\delta'|k|^4, \tag{5.13}$$

for $|k|\le 1$. This bounds the first term in (5.10).

Using (4.25, 4.26) to bound the derivative of $\hat{\mathcal{T}}'$, (5.8) and (5.6) imply

$$|\hat{r}(k)|\le C(L)\epsilon^2\delta \tag{5.14}$$

for $|k|\le 2$ (note that we apply (4.25, 4.26) to $n\ge 1$ here, i.e. we can assume that $r^2L^{1/4}$ is large enough). Note that $\hat{r}(k)$ satisfies $\hat{r}(k)=-\zeta k^2+\mathcal{O}(k^4)$ so that we infer from (5.5) $\hat{\beta}(k)+\hat{r}(k)=\mathcal{O}(|k|^4)$. Combining this with (5.7), (5.14) and a Cauchy estimate yields

$$|\hat{\beta}(k)+\hat{r}(k)|\le C(L)\epsilon^2\delta|k|^4\le\frac{1}{2}\epsilon\delta'|k|^4, \tag{5.15}$$

for $|k|\le 1$. Then, (5.13) and (5.15) imply (4.29).

Next, we prove (4.30). Combining (4.25, 4.26) with $n\ge 1$ and (5.8, 5.6) with (5.11), we infer

$$|r(x)|\le C(L)\epsilon^2\delta e^{-2|x|}\le\frac{1}{4}\epsilon\delta' e^{-2|x|}. \tag{5.16}$$

As for $\tau$, we have from (5.12),

$$\tau(x)=L^d\sum_{m=1}^{L^2}\binom{L^2}{m}(\mathcal{T}^{L^2-m}(\cdot/\rho)t^m)(Lx).$$

Consider first the $m=1$ term. Its Fourier transform is given by the $m=1$ term in (5.12). By shifting the integration contour the $m=1$ is thus bounded by

$$L^2e^{-2|x|}\int\left|\hat{t}\left(\frac{k}{L}\right)\hat{\mathcal{T}}\left(\rho\frac{k}{L}\right)^{L^2-1}\right|dk,$$

where $|\Im k|=2$. Use (4.29) for $|k|\le L$, and (4.25), (4.26) (for $n\ge 1$, since $t_0=0$) to bound the integral over $|k|\le L$ by $C\epsilon\delta L^{-4}$. For $|k|\ge L$, we use the fact that, by (4.30), $|\hat{t}(\frac{k}{L})|$ is bounded by $C\epsilon\delta$, and that, by (4.25), (4.26), the integral of $\hat{\mathcal{T}}(\rho\frac{k}{L})^{L^2-1}$ over $|k|\ge L$ is less than $C\exp(-cL^{\frac{1}{2}})$. Hence altogether the $m=1$ term is bounded by

$$C(L^{-2}+e^{-cL^{1/2}})\epsilon\delta e^{-2|x|}.$$

The $m\ge 2$ terms in (5.16) are easily bounded, using (4.25, 4.26) and (4.30) and only add $\delta$ to $(L^{-2}+e^{-cL^{1/2}})$. Hence $\tau(x)$ is bounded by the right-hand side of (5.16). Combining these bounds with (5.3), (4.30) follows for $t'$. $\square$

Now, the proof of Proposition 2 is straightforward:

*Proof of Proposition 2* We get (2.19) by combining (4.25, 4.26) for $n \geq 1$ and (4.30). To show (2.20), we define $D = \lim_{n\to\infty} \rho_n^2 D_0$, and we have:

$$T_n(x) - T_D^*(x) = \mathcal{T}_n\left(\frac{x}{\rho_n}\right) - T_D^*(x) + t_n(x).$$

From (5.8), (5.6), we get that (for $\epsilon$ small) $|\rho_n^2 D_0 - D| \leq \delta_n$. Then, we use (4.25, 4.26) and bound the derivative of $\mathcal{T}_n(k)$ to get (2.20) for the first term. We use (4.30) for the second. Finally, we get (2.21) from (4.28). □

## 6 Proof of Theorem 1

*Proof* Since the functions $f_i$ and the paths $\omega$ are continuous, it is enough to prove Theorem 1 for any given family $\mathbf{f} = (f_i)_{i=1}^{\kappa}$ and for all sets of times $\mathbf{t} = (t_i)_{i=1}^{\kappa}$, where $t_i$ belongs to the dense set $\bigcup_\ell L^{-2\ell}\mathbb{N}$. So, let $t_i \in L^{-2\ell}\mathbb{N}$, $i = 1, \ldots, \kappa$, and $\ell$ fixed. We use (2.5) and $p_m = T_m + b_m$ as in (4.1) to get (recall that $n = \ell + m$)

$$\mathcal{E}_n \prod_i f_i(\omega(t_i)) = \sum_A \int dz dv' K_{mA}(z', z) b_{mA}(z) \prod_{i=1}^{\kappa} f_i(L^{-\ell} x_i) := \sum_A I_n(A, \mathbf{f}, \mathbf{t}, \ell) \quad (6.1)$$

where notation is as in (4.2) with $L$ replaced by $L^\ell$, $v_0 = 0$ and $v_{l+1} = v'$. The $x_i$'s form a subset of the $u_j, v_j$'s. It is useful to remember that the product over $T$ and $b$ in (6.1) is ordered over the time interval $[0, L^{2\ell}]$.

The terms $I_n(A, \mathbf{f}, \mathbf{t}, \ell)$ are random variables. We show first that, for any $\mathbf{f}$, there is a set $\mathcal{B}$ of measure one such that, for $b \in \mathcal{B}$, $\lim I_n(A, \mathbf{f}, \mathbf{t}, \ell) = 0$, for all $A \neq \emptyset$ and all $\mathbf{t}$.

First note that, if $f$ is polynomially bounded, then, for all $\gamma > 0$, we can find a constant $C(\gamma, f)$ such that $|f(x)| \leq C(\gamma, f)\exp(\gamma|x|)$. Thus writing $f_i = C(\gamma, f_i) f_i'$ it is enough to prove the claim for $f_i$ such that $C = 1$; $\gamma$ will be chosen below.

Next, since $v_0 = 0$ and $x_i$ is one of the $u_j, v_j$,

$$|x_i| \leq \sum_{j=1}^{\ell} (|v_{j-1} - u_j| + |u_j - v_j|). \quad (6.2)$$

Therefore, writing $I_n(A, \mathbf{f}, \mathbf{t}, \ell) = I_n(A)$,

$$\langle I_n(A)^2 \rangle \leq \int dz dv' \tilde{K}_{m\mathcal{A}}(z', z) |\langle \tilde{b}_{m\mathcal{A}}(z) \rangle| \quad (6.3)$$

where, in $\tilde{K}_m$, $T_m(u)$ is replaced by $e^{\kappa\gamma|u|} T_m(u)$ and $b_m(t, u, v)$ by $e^{\kappa\gamma|u-v|} b_m(t, u, v)$ and $\mathcal{A} = A \sqcup A$ (note that there are twice as many variables $z$ and $v'$ compared to (6.1)).

Next, expand the expectation value in (6.3) in terms of connected correlation functions, using (4.21). We need to bound then

$$J := \int dz dv' \tilde{K}_{m\mathcal{A}}(z', z) \prod_{B \in \Pi} |\langle \tilde{b}_{mB}(z) \rangle^c| \quad (6.4)$$

where $\Pi \in \mathcal{P}(\mathcal{A})$, since the number of terms in (4.21) depends on $|A|$, i.e. on $L^{2\ell}$. By (1.10), $\sum_{t_i \in B} |u_i - v_i| \le \tau_B(z)$. Thus for $\kappa\gamma < \lambda/2$,

$$J \le \int dz dv' \tilde{K}_{m\mathcal{A}}(z', z) \prod_{B \in \Pi} e^{\frac{1}{2}\lambda\tau_B(z)} |\langle b_{mB}(z)\rangle^c|. \tag{6.5}$$

Note that each time appears in $\mathcal{A}$ at most twice. If there are factors with $d(B) \ne 0$, we use (2.17) and the remaining integrals consist of order $\ell$ convolutions of $\tilde{T}$, each of which is bounded by $\|\tilde{T}\|_1 \le C$, by Lemma 2, for $\kappa\gamma < 2\lambda/L$ (see (3.9)). If only $B$'s with $d(B) = 0$ occur, use $\|\tilde{T}\|_1 \le C$ for all the factors $\tilde{T}$ occurring after the last $B$ and use (2.18) for that one (we necessarily have an integral over $v$ here, since we integrate over the last $v$ variable, denoted $v'$ in (6.1)). The result is:

$$\langle I_n(A)^2\rangle \le C(\ell, L)\epsilon^{2|A|}\delta_m. \tag{6.6}$$

By Chebyshef's inequality we get

$$P(|I_n(A)| > 1/k) \le C(\ell, L)k^2\epsilon^{2|A|}\delta_m. \tag{6.7}$$

Since, by (2.16), $\sum_m \delta_m < \infty$, we get, by the first Borel-Cantelli lemma, that, for any given $\mathbf{f}, \mathbf{t}$, $A \ne \emptyset$ and $k \in \mathbb{N}$, there is a set of measure one, $\mathcal{B}_k(\mathbf{f}, \mathbf{t}, A)$, on which $\limsup_n |I_n(A, \mathbf{f}, \mathbf{t}, \ell)| \le 1/k$. Since the number of sets $A$ in (6.1) is finite, given $\ell$, and since the set of sequences $\mathbf{t}$, with $t_i \in \bigcup_\ell L^{-2\ell}\mathbb{N}$, is countable, $\mathcal{B}(\mathbf{f}) := \bigcap_{\mathbf{t}} \bigcap_{\mathbf{A}\ne\emptyset} \bigcap_{\mathbf{k}} \mathcal{B}_{\mathbf{k}}(\mathbf{f}, \mathbf{t}, \mathbf{A})$ is a set of measure one on which

$$\lim_{m\to\infty} \left(\mathcal{E}_{m+\ell} \prod_i f_i(\omega(t_i))\right) - E_\ell^{T_m} \prod_i f_i(L^{-\ell}\omega(L^{2\ell}\omega(t_i))) = 0 \tag{6.8}$$

where, as we recall from Sect. 2 (see (2.5)), $E_\ell^{T_m}$ is the expectation in the random walk with transition probability $T_m$, in time $L^{2\ell}$; thus, the second term in (6.8) corresponds to the $A = \emptyset$ term in (6.1).

We are left with proving a deterministic statement, namely that the second term in (6.8) converges to $\mathcal{E}^D \prod_i f_i(\omega(t_i))$. Let again $t_i \in L^{-2\ell}\mathbb{N}$. Then,

$$\mathcal{E}^D \prod_i f_i(\omega(t_i)) = E_\ell^{T_D^*} \prod_i f_i(L^{-\ell}\omega(L^{-2\ell}t_i)). \tag{6.9}$$

Write $T_m = T_D^* + \tau_m$. Bounding the $f_i$'s as above, see (6.2), we get that the difference between the second term in (6.8) and (6.9) is bounded by

$$\sum_{k=1}^{N} \binom{N}{k} \int dx \, ((\tilde{T}_D^*)^{N-k}\tilde{\tau}_m^k)(x)$$

where $N = L^{2\ell}$ and the tilde is defined as above. By (2.20) and the explicit form (2.15) of $T_D^*$, this sum is bounded by $C(\ell, L)\delta_m$ and the claim follows. □

*Remark 6* The main estimate that one needs, in order to extend this result to $\mathcal{P}$-almost sure weak convergence, is to show that the sequence $\nu_{\mathbf{T}}$ is almost surely tight. To prove this, it is enough, by standard results (see e.g. [1], Theorem 12.3) to show that there exists a set $\mathcal{B}$ with $\mathcal{P}(\mathcal{B})=1$, such that $\forall \mathbf{b}\in\mathcal{B}$, $\exists K$, such that, $\forall t,s\in[0,1]$,

$$\int d\nu_T(\omega)(\omega(s)-\omega(t))^4 \le K|s-t|^2. \tag{6.10}$$

To prove the latter estimate, consider first finite range walks, i.e. such that $p(t,u,v)=0$ if $|u-v|\le R$, for some $R<\infty$ (the extension to exponentially decaying walks is possible but requires still more work).

Then we write $[\mathbf{T}(t-s)]=\sum_{k=0}^{n}\tau_k L^{2k}$, where $T\le L^{2(n+1)}$, $\tau_k\le L^2-1$, and write $(\omega(s)-\omega(t))^4$ as a sum over contributions coming from the various scales $k=0,\dots,n$. To bound each of these contributions, we need to bound the probability of large jumps.

To simplify notations, consider $\mathbf{T}=L^{2n}$, $s=0$, and thus $\omega(s)=0$. Write

$$\omega(t)=L^{-n}\sum_{\ell=0}^{n}(\Delta\omega)_\ell L^\ell, \tag{6.11}$$

where $(\Delta\omega)_\ell$ is the increment of the renormalized walk on scale $\ell$ (whose distribution is given by $p_\ell$) corresponding to the time interval $\tau_\ell L^{2\ell}$. Now assume that: $\exists C<\infty$ such that, $\forall\ell\ge\frac{n}{4}$, $\forall t\le L^{2n}$, $\forall u$, $|u|\le RL^{2n}$,

$$\int dv p_\ell(t,u,v)|u-v|^4\le C. \tag{6.12}$$

Since the walks are of finite range $R$, we have

$$|(\Delta\omega)_\ell|L^\ell\le RL^{2\ell}$$

i.e.

$$(|\Delta\omega_\ell|L^\ell)^4\le RL^{2n}$$

for $\ell\le n/4$.

So, under the previous assumptions, and using the fact that the sum over $\ell$ in (6.11) is geometric, we get

$$\int d\nu_T\omega(t)^4\le C(L)(RL^{-2n}+L^{4(m-n)})$$

where $m=\max\{\ell|\tau_\ell\ne 0\}$. On the other hand, by definition of $m$, $|t-s|=t\ge L^{2m-2n}$. Since $t>0$ is fixed, $n-m$ is bounded as $n\to\infty$, and thus, $L^{4(m-n)}$ is larger than $RL^{-2n}$. So, under assumption (6.12), we get

$$\int d\nu_T\omega(t)^4\le C(L)t^2, \tag{6.13}$$

as required.

Since $p_\ell=T_\ell+b_\ell$, the bound (6.12) holds if we assume that

$$\forall\ell\ge\frac{n}{4},\ \forall t\le L^{2n},\ \forall u,\ |u|\le RL^{2n},\quad \int dv|b_\ell(t,u,v)|e^{\frac{\lambda}{2}(u-v)}\le 1. \tag{6.14}$$

To prove this, we need to extend the bounds in Proposition 1 so that we have, in their right-hand side, an $n_0$ dependent power of $\delta_n$, when $A$ contains $n_0$ copies of the same times.

This is not difficult. Only the iteration of the case $d(A')=0$, done after (3.23), requires more care.

Then, we can use an inequality like (6.6, 6.7) (with 2 on the left-hand side replaced by $n_0$ and $\delta_m$ replaced by a sufficiently large power of $\delta_m$) and (2.16), to prove that (6.14) holds, for given $\ell, t, u$, with probability larger than $1-CL^{-cn_0\ell}$.

Now, since the number of variables $t$ and $u$ is of order $L^{2nc'}$, for some $c'$, and since $\ell \geq \frac{n}{4}$, we have that (6.14) holds with probability $1-L^{-\mathcal{O}(n)}$, for all $t$, $u$ considered, if we choose $n_0$ large enough. Now, letting $\mathcal{B}=U\mathcal{B}_m$ where $\mathcal{B}_m$ is the set of environments such that (6.14) holds for all $n \geq m$, one get $\mathcal{P}(\mathcal{B})=1$, and, for every environment in $\mathcal{B}$, (6.13) holds for all $\mathbf{T}$ large enough.

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